Fourier series and the Gibbs phenomenon

William J. Thompson

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27599-3255

(Received 11 June 1991; accepted 18 October 1991)

The occurrence in Fourier series of an overshoot effect near function discontinuities, called the Gibbs phenomenon, is discussed from a pedagogical viewpoint. The reader is led along a path to discover why the phenomenon depends only upon the existence of the discontinuity, but not on other properties of the function that is Fourier analyzed.

I. INTRODUCTION

An understanding of Fourier series and their generalizations is important for physics and engineering students, as much for mathematical and physical insight as for applications. Students are usually confused by the so-called Gibbs phenomenon—the persistent discrepancy, an “overshoot,” between a discontinuous function and its approximation by a Fourier series as the number of terms in the series becomes indefinitely large. Although the phenomenon is mentioned under Fourier series in almost every textbook of mathematical physics,¹ the treatment is often confined to the square-pulse example, so that students are often left wondering what aspect of this pulse gives rise to the phenomenon and whether it depends upon the function investigated. The subject has also been discussed in this Journal,²⁻⁴ but only from a limited perspective. The aim of this paper is to lead the reader along the steps to solving the mystery of the Gibbs phenomenon.

Historically, the explanation of the Gibbs phenomenon is usually attributed to one of the first American theoretical physicists, J. Willard Gibbs, in two notes published in 1898 and 1899 (Ref. 5). Gibbs was motivated to make an excursion into the theory of Fourier series by an observation of Albert Michelson that his harmonic analyzer (one of the first mechanical analog computers) produced persistent oscillations near discontinuities of functions that it Fourier analyzed, even up to the maximum harmonic (80) that the machine could handle. Examples of these oscillations are shown in the 1898 paper⁶ by Michelson and Stratton. The phenomenon had, however, already been observed and essentially explained by the English mathematician Henry Wilbraham 50 years earlier⁷ in correcting a remark by Fourier on the convergence of Fourier series. It might be more appropriate to call it the Wilbraham–Gibbs phenomenon than the Gibbs phenomenon.

The first extensive generalization of the Gibbs phenomenon, including the conditions for its existence, was provided by the mathematician Bôcher in 1906 (Ref. 8). Both this treatment and those in subsequent mathematical treatises on Fourier series⁹ are at an advanced level, usually unsuitable for physics students. In the following we consider by a method accessible to physics and engineering students the problem of Fourier series for functions with discontinuities. The method is rigorous; it contains the essence of the mathematical treatments without their complexity; and it discusses how to estimate the overshoot numerically.

The presentation is organized as follows. In Sec. II a generalization of the sawtooth function is made, to include
in a single formula the conventional sawtooth, the square-pulse, and the wedge functions. Their Fourier amplitudes can be calculated readily from the Fourier-series formula that we then derive. This formula provides the starting point for understanding the Gibbs phenomenon in Sec. III. The Appendix gives enough detail on the numerical methods used to estimate the overshoot values that students can readily make the calculations themselves.

II. FOURIER SERIES FOR THE GENERALIZED SAWTOOTH

The generalized sawtooth function that we introduce is sketched in Fig. 1 (a) and is defined by

\[ y(x) = 1 - ((\pi - x)s_L, \quad 0 < x < \pi, \]

\[ = 1 - D + (x - \pi)s_R, \quad \pi < x < 2\pi, \quad (1) \]

in terms of the slopes on the left- and right-hand sides of the discontinuity \( s_L \) and \( s_R \) and the extent of the discontinuity \( D \). From this definition we obtain several functions whose Fourier series are commonly investigated by physicists and engineers, namely; the square pulse has \( s_L = s_R = 0, \quad D = 2 \); the wedge that starts and ends with \( y = 0 \) has \( s_L = -s_R = 1/\pi, \quad D = 0 \); the conventional sawtooth with the same end points has \( s_L = s_R = 1/\pi, \quad D = 2 \). There is also an implied discontinuity of \( \pi(s_R - s_L) = D \) at \( x = 0 \) or \( 2\pi \), which will produce another overshoot. It is sufficient for our discussion to consider only the discontinuity at \( x = \pi \).

A function and its Fourier amplitudes are related by

\[ y(x) \approx y_M(x), \quad (2) \]

where the Fourier series to \( M \) terms is

\[ y_M(x) = \sum_{k=0}^{M} \left[ a_k \cos(kx) + b_k \sin(kx) \right]. \quad (3) \]

For our purposes the Fourier amplitudes \( a_k \) and \( b_k \) can most conveniently be obtained from

\[ a_k = \text{Re}[c_k]/(1 + \delta_{k,0}), \quad b_k = \text{Im}[c_k] \quad (4) \]

in terms of the complex Fourier coefficients \( c_k \) given by

\[ c_k = \int_{0}^{2\pi} dx \, y(x) e^{-ikx}. \quad (5) \]

Recall that the Fourier amplitudes, for given \( M \), provide the best fit (in the least-squares sense) of an expansion of the type Eq. (3) to the function \( y \). Therefore, attempts to smooth out the Gibbs phenomenon by applying damping factors necessarily worsen the overall fit. This fact is often forgotten and needs to be recalled. The Appendix gives enough detail on the numerical methods used to estimate the overshoot values that students can readily make the calculations themselves.

Fig. 1. The generalized sawtooth function, Eq. (1), used to discuss the Gibbs phenomenon. The solid lines in (a) show the function and in (b) they show its derivative with respect to \( x \).
series value at the discontinuity is just the average of the function values just to the left and right of the discontinuity, namely zero.

(3) The series in Eq. (11) is just twice the sum over the reciprocal squares of the odd integers not exceeding \( M \). Therefore, it increases uniformly as \( M \) increases. This convergence is illustrated for the wedge function in Fig. 2. The limiting value of the series can be obtained in terms of the Riemann zeta function, as \( 3\zeta(2)/2 = \pi^2/4 \). The Fourier series then approaches

\[
y_\omega(\pi) = a_0 - \pi(s_R - s_L)/4 = 1 - D/2.
\]

Thus, independently of the slopes on each side of the discontinuity, the series tends to the average value across the discontinuity—a commonsense result.

Notice that in the above we took the limit in \( x \), then we examined the limit of the series. The result is perfectly reasonable and well behaved. The surprising fact about the Gibbs phenomenon, which we now examine, is that taking the limits in the opposite order produces quite a different result.

III. THE GIBBS PHENOMENON

Now that we have examined a function with no discontinuity in value but only a discontinuity in slope, we direct our steps to the study the value discontinuity. Consider any \( x \) which is not at a point of discontinuity of \( y \). The overshoot function, defined by,

\[
O_M(x) = y_M(x) - y(x)
\]

is then well defined, because both the series representation and the function itself are well defined. As we derived at the end of the last section, if we let \( x \) approach the point of discontinuity with \( M \) finite, we get a commonsense result. Now, however, we stay near (but not at) the discontinuity. We will find that the value of \( O_M \) depends quite strongly on both \( x \) and \( M \). In order to distinguish between the usual oscillations of the series approximation about the function and a Gibbs phenomenon, one must consider the behavior of \( O_M \) for large \( M \) and identify which parts (if any) persist in this limit.

The signs in these definitions are chosen so that the functions are positive near \( x = \pi \).

One may use Eqs. (14) through (17) to investigate the overshoot values directly as a function of the number of terms in the Fourier series \( M \) and the values of \( x \) near the discontinuity at \( x = \pi \). The Appendix has a discussion of some numerical procedures for this. It is also interesting to see what progress can be made analytically. In particular, for what value of \( x \), say \( x_M \), does \( O_M(x) \) have a maximum for \( x \) near \( \pi \)? To investigate this question, let us do the obvious and calculate the derivatives of the terms in Eq. (14), which requires the derivatives of the trigonometric series, Eqs. (15)-(17). We have

\[
\frac{dS_+}{dx} = 2 \sum_{k=1}^{M} \cos(kx),
\]

\[
\frac{dS_D}{dx} = \sum_{k=1}^{M} (1 - \pi_k) \sin(kx).
\]

To evaluate the latter two series in closed form, we write the cosine as the real part of the complex-exponential function, then recognize that one has geometric series in powers of \( \exp(ix) \), which can be summed by elementary means. Thus

\[
\frac{dS_-}{dx} = S_D,
\]

\[
\frac{dS_+}{dx} = 2 \sum_{k=1}^{M} \cos(kx),
\]

\[
\frac{dS_D}{dx} = \sum_{k=1}^{M} (1 - \pi_k) \sin(kx).
\]

Since \( S_D \) is not known in closed form, there is probably no simple way to evaluate the derivative of \( S_- \) in closed form. It will turn out that we do not need it subsequently. Collecting the pieces together, we finally have the result for the derivative of the overshoot function at any \( x < \pi \).
\[ \frac{dO_M}{dx} = (s_R - s_L) \left( \frac{1}{2} - \frac{1}{\pi} \frac{dS}{dx} \right) \]
\[ - \frac{(s_R - s_L)}{2} \frac{\sin[(M + 1)x]}{\sin(x/2)} \]
\[ + \frac{D \sin[(M + 1)x]}{2 \sin(x/2)} \]  

This derivative is apparently a function of the independently chosen quantities \( s_R, s_L, D, \) and \( M. \) Therefore, the position of the overshoot extremum (positive or negative) seems to depend upon all of these.

The only example that we considered in Sec. II that had \( s_R \neq s_L \) was the wedge, which was well behaved near \( x = \pi \) because \( D = 0. \) Figure 2 shows the overshoot function for the wedge, for three small values of \( M, \) namely, 1, 3, and 15. For the wedge only the \( S_- \) series, Eq. (15), is operative for \( O_M, \) and this series converges as \( 1/k^2, \) rather than as \( 1/k \) for the other series. Note the very rapid convergence, which improves about an order of magnitude between each choice of \( M. \) Clearly, there is no persistent overshoot.

For the square-pulse function, the common example for displaying the Gibbs phenomenon, \( s_R = s_L = 0, \) so that in Eq. (23) the extremum closest to (but not exceeding) \( \pi \) will occur at

\[ x_M = \pi - \pi/(M + 1). \]  

By differentiating the last term in Eq. (23) once more, it will be found by the conscientious reader that the second derivative is negative at \( x_M, \) so this \( x \) value produces a maximum of \( O_M. \) Indeed, from Eq. (22) it is straightforward to predict that there are equally spaced maxima below \( x = \pi \) with spacing \( 2\pi/(M + 1). \) Thus the area under each excursion above the line \( y = 1 \) must decrease steadily as \( M \) increases.

The square-pulse overshoot behavior is shown in Fig. 3, in which we see the positive overshoot shrinking proportionally closer to \( \pi \) as \( M \) increases. By making numerical calculations, as described in the Appendix, students will discover that the extent of the overshoot is also remarkably independent of \( M \) for the values shown, namely \( O_M(x_M) = 0.179 \) to the number of figures shown. One way to estimate this well-known result for large \( M \) is given in the Appendix. The fact that \( O_M \) converges to a nonzero value for large \( M \) identifies this as a genuine overshoot, as discussed in the first paragraph of this paper and below the defining Eq. (13).

The sawtooth function is our second example for the Gibbs phenomenon. For this function we have \( s_R = s_L = 1/\pi, \) and \( D = 2. \) If these values are inserted in Eq. (23) we find exactly the same position for the location of the maximum overshoot, namely that given by Eq. (24). The amount of the overshoot also tends to the same magic value, 0.179, as one will discover by investigating the problem numerically, as suggested in the Appendix. Other properties, such as the locations of the zero-overshoot positions on either side of \( x_M, \) will be found to depend on the parameters determining the shape of \( y(x). \)

So, what's going on here, and have we been led along a primrose path? According to Eq. (23), it looks as if the value of \( x_M \) should depend on the shape of the function whose Fourier series we determine. But we discovered that, if there is a discontinuity \( (D \neq 0), \) there is no dependence of its position or of the overshoot value on the shape of the function, at least for the two examples that we investigated.

The way to solve this puzzle is to take a different view of the original function \( y(x), \) for example as given by Eq. (1). We may consider this as a linear superposition of a function with no discontinuities in its values, plus a constant (say unity) broken by a drop of amount \( D \) at \( x = \pi. \) Since taking Fourier series is a linear transformation, in that sums of functions have Fourier amplitudes which are the sums of the amplitudes from each series separately, only the discontinuous part gives rise to an overshoot in the limit of large \( M. \) Then the overshoot is at \( \pi/(M + 1) \) below \( x = \pi, \) since it arises only from the discontinuity. Because of the symmetry of the discontinuity, there will be a mirror undershoot in the opposite direction just above \( x = \pi. \)

The terms in Eq. (23) that do not involve \( S_R \) are just oscillations of the series about the function. These oscillations damp out for large enough \( M, \) as we saw for the wedge at the end of Sec. II. A very asymmetric wedge, having \( D = 0, \) \( s_+ > 0, \) \( s_- < 0, \) but \( |s_R| \geq |s_L|, \) may be used to very nearly reproduce the effects of a discontinuity. Such discontinuities of derivatives produce slow convergence of the series, but they cannot cause the overshoot.

Our discussion is now complete: We conclude that, as soon as we have studied the square-pulse function that is all there is to understand about the Gibbs phenomenon in Fourier series.

**APPENDIX: NUMERICAL METHODS**

The numerical methods for investigating the Gibbs phenomenon in Fourier series are straightforward, but require care if a large number of terms \( M, \) is used in the series. This is because for \( x \) near \( \pi, \) summation over terms containing sine or cosine of \( kx \) oscillate in sign very rapidly as \( k \) or \( x \) changes.

I found that with the precision of my computer the values of \( S_\pm \) and \( S_R \) were calculated accurately for \( M \) up to about 300. Past this, even with the use of double-precision computer arithmetic, unreliable results were produced. In order to make this claim, an alternative estimate of the sum is needed. Fairly elegant methods for summing the remainder series and thereby estimating bounds on the overshoot are given by, for example, Sommerfeld and by Solymar. We provide a simplified treatment as follows.

The defining Eq. (17) for \( S_R \) can be first replaced by
twice the summation over only odd values of \( k \), then this sum can be approximated by an integral if \( M \) is large. The result can be converted to the integral

\[
S_D(x) \approx \int_{0}^{(M-1)x'} \frac{\sin t}{t} \, dt, \tag{25}
\]

where \( x' = \pi - x \). Since we want \( S_D \) evaluated at the peak value nearest to \( \pi \), we set \( x \) to \( x_M \) given by Eq. (24). Then, within a correction of order \( 1/M \), the resulting integral is to be evaluated with an upper limit of \( \pi \). The integral in Eq. (25) can easily be done by expanding the sine function in its Maclaurin series, dividing out the \( t \), then integrating term by term, to obtain

\[
S_D(x_M) \approx \int_{0}^{\pi} \frac{\sin t}{t} \, dt = \pi \left( 1 - \frac{\pi^2}{3!} + \frac{\pi^4}{5!} - \frac{\pi^6}{7!} + \cdots \right). \tag{26}
\]

When the value of this integral summed out to \( \pi^{10} \) is substituted in the overshoot Eq. (14) for the square pulse, we find that \( O_M(x_M) = 0.179 \) to three significant figures, in complete agreement with direct calculation of the series overshoot for both the square pulse and the sawtooth found in Sec. III.

\[\text{References:}\]